Archimedes: Equilibrium of the plane, Book I

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1 Comments on the debate about Book I

Archimedes' text is poised between physics and mathematics and it is difficult to categorize it.

(1) Is it *pure mathematics*? (where, for example, the concepts baros, equilibrium, etc. are being given their "starter axioms") (2) Is it *physics*? (where the mathematics merely has a descriptive, rather than ontological, purpose) (3) Is it some curious combination of mathematics and physics? (where the concepts are on the road to being imported into math, starting their lives in physics, the way—say—Maxwell's equations move into Riemannian geometry)

If (1), then the issue of circularity is gone: all of math is circular and that's a good thing. If (2), then—taking Mach's viewpoint of physical law as as being (nothing more than) the most economical way of expressing the sum total of our observational experience—hey, if your experiments were specifically designed to verify Archimedes' postulates, and you want to know whether they are also evidence for his law of the lever, well, you have to use his derivation of the law of the lever from the postulates to see this; if, however, you experiments were designed to check the law of the lever, then running his derivation backwards gives you evidence for his postulates. What's the complaint? (Answer: some inner intent.) If (3) then one is in that middle-zone, and the derivations are even more useful to convince you of some inner coherence of the set-up.

2 Law-of-the-lever

Here is a proof—perhaps *the* only proof[1] based on the postulates of Archimedes that Archimedes' law-of-the-lever holds.

Imagine a homogeneous bar of length $2m+2n$, placed symmetrically on a balance line. So a "weight" $m+n$ is distributed to the left of the fulcrum, and an equal weight is distributed to the right. Now make a laser-like slicing of the bar, splitting it into two bars of weights $2m$ (say, on the left) and $2n$ (on the right). Replace each of those bars by single weights $2m$ and $2n$ concentrated at their

[1] given with minor modifications in many of our readings

respective centers of gravity. Then the concentrated weight weighing $w := 2m$ is of distance $d := n$ to the left of the fulcrum while the concentrated weight weighing $w' := 2n$ is of distance $d' := m$ to the left of the fulcrum. By the postulates of Archimedes, we get that this is in equilibrium.

Note that:

$$(*) \quad w \cdot d \; = \; w' \cdot d'.$$

Using further postulates of Archimedes, the two weights w, w' on either side of a fulcrum and of distances d, d' (respectively) to the fulcrum are in equilibrium if and only if (*) holds. This is his Law-of-the-lever.

Now this is 'proved' by Archimedes, subject to his postulates. Mach has interesting comments about this proof, one of which I will paraphrase, putting it in slightly more mathematical terms than he does, but without entirely violating his intent, I think.

1. Some *unstated* pre-postulates are required to hold, before one can even begin to make the above kind of argument. For example,
   - There *is* a property that determines equilibrium: for example, there a single function

     $$F(\ldots \text{relevant variables} \ldots)$$

     such that if the function F evaluated on the state of the left-of-the-fulcrum part of the balance bar is equal to the function F evaluated on the state of the right-of-the-fulcrum part of the balance bar, then (*and only then*) is equilibrium achieved.
   - The "*relevant variables*" referred to above are just weight w and distance-to-fulcrum d (and not, say, dependence on the position of the observer, or color of the balance bar). So our function F may be thought of as a two-variable function $F(w, d)$.
2. The stated postulates are the ones at the beginning of Archimedes' treatise.

Now Mach said something in the initial edition of *Science of Mechanics* that stirred up quite a debate, and deserves some discussion. The issue has to do with the fact that

- the statement that the function $F(w, d)$ is linear in both variables, and therefore—up to scaling— $F(w, d) = w \cdot d$ is equivalent to Archimedes' law of the lever, and
- this linearity follows from the axioms, and
- yet Mach worries that people think they are "making up properties of nature with the help of self-evident suppositions." (See page 27; also Page 19 where Mach curiously uses the word "covert").

This has inspired much argument; recently Palmieri. My feeling about Palmieri is that he is very heavy-handed in his derivations, so here is what is a more direct derivation of bilinearity of $F(w, d)$.

Given Archimedes' axioms we get that

$$(\mathbf{2}) \quad F(w_1 + w_2, d) \ = \ F(w_1, d) \ + \ F(w_2, d),$$

(so it is linear in the first variable) and:

$$(\mathbf{3}) \quad F(2w, d) \ = \ F(w, d - \Delta) \ + \ F(w, d + \Delta).$$

Putting **(2)** and **(3)** together we can write

$$(\mathbf{4}) \quad F(2w, d) \ = \ F(w, d) + F(w, d) \ = \ F(w, d - \Delta) \ + \ F(w, d + \Delta).$$

So

$$(\mathbf{5}) \quad F(w, d + \Delta) - F(w, d) \ = \ F(w, d) - F(w, d - \Delta),$$

or changing notation to make it easier to look at: for any fixed w put $f(x) = F(w, x)$ and set $a := d - \Delta$ and $b := \Delta$, so we have for *any* $a, b \geq 0$:

$$(\mathbf{6}) \quad f(a + b) - f(a) = f(a + 2b) - f(a + b)$$

and iterating this as many times as we want, we get:

$$(\mathbf{7}) \quad f(a + b) - f(a) = f(a + 2b) - f(a + b) = f(a + 3b) - f(a + 2b) = \dots$$

or, since our f vanishes at 0, we get (taking $a = 0$) that

$$(*) \qquad f(nb) = nf(b)$$

for any b and positive integer n. Or, putting $b = c/m$ we can also write this as $f(c/m) = 1/mf(c)$ for any c and positive whole number m, and putting this together by taking c to be nx for any $x > 0$ and any positive whole number n we get $f(nx/m) = nf(x/m) = n/m \cdot f(x)$; i.e.,

$$(\mathbf{8}) \quad f(rx) = rf(x)$$

for any x and positive rational number r. And so (at least under the assumption that f is continuous[2]) f is linear.

Why does Mach think that there is something circular in this?

[2]This is a it necessary assumption; it is false without it.